Unified Algebraic Approach to Approximation of Lateral–Directional Modes and Departure Criteria

Giovanni Mengali*
University of Pisa, 56122 Pisa, Italy
and
Fabrizio Giulietti†
University of Bologna, 47100 Forlì, Italy

Introduction

LITERAL expressions involving aerodynamic derivatives and inertial data provide physical insight into the behavior of the aircraft motion. For instance, they are well suited for the optimization of flight control systems, both for the setup of nominal system design and for the prediction of motion characteristics in off-nominal problems.[1] In fact, such literal expressions show the direct connection between the transfer function poles and zeros as a function of the relative values of certain key derivatives and give valuable insight into the nature of the associated control problem. More precisely, they show the detailed effect of particular stability derivatives on aircraft characteristics. Also, they are of fundamental importance in obtaining stability derivatives from flight data and in developing analytical departure criteria to be used in the design process of new aircraft configurations.[2]

However, it is difficult to find exact closed-form solutions, especially to the aircraft modes, because both the longitudinal and the lateral aircraft eigenvalues come from fourth-order characteristic equations. Therefore, an approximation is required to arrive at reasonably compact and usable expressions that delineate the dominant effects. In many cases this process of approximation is useful for the designer. Indeed, the omission of certain terms, which are relatively unimportant, allows such important simplifications to be made that the relation between cause and effect becomes apparent. Very often such effects vary among vehicle types, so that literal approximate factors, which apply to all vehicles for all flight conditions, are quite difficult to obtain. This conclusion is particularly apparent when aircraft lateral–directional modes are considered. In particular, an accurate expression for the Dutch-roll damping is traditionally known to be a difficult task, although relatively satisfactory approximations for the spiral, roll, and Dutch-roll natural frequency are available, for example, see McRuer et al.,[3] pp. 367–377. This is a serious difficulty when analytical approximations to aircraft departure criteria are sought. In fact, a natural approach to this problem is to set to zero the Dutch-roll damping expression to predict its instability. However, this is impractical as long as the Dutch-roll damping approximation is inaccurate. The main reason for this difficulty comes primarily from the physical behavior of lateral–directional motion. In fact, the easiest way to obtain accurate simplified models is to resort to a timescale analysis of the problem, separating slow from fast modes. However, when applied to the lateral–directional motion, this approach is not fully satisfactory because the complete fourth-order equations of motion are approximated through a first-order system (the spiral dynamics) and a third-order system (the roll and Dutch-roll dynamics). In fact the roll and Dutch-roll modes are strictly connected, being the ratio between roll and Dutch-roll frequencies typically in the range 0.5–2 for most aircraft and flight conditions.[4] This explains why traditional Dutch-roll approximations are unsatisfactory inasmuch as literal expressions to the poles of third-order systems are difficult to obtain.

Starting from this viewpoint, in this Note we propose a new methodology to obtain accurate analytical expressions for the lateral–directional modes that are well suited to predict the occurrence of dynamic instabilities.

Algebraic Approximations

To develop approximations of lateral–directional modes, consider the standard factorization of the characteristic polynomial,

$$P_{\text{lat}}(s) = \left(s^2 + 2\zeta_{\text{DR}}\,\omega_{\text{DR}}s + \omega_{\text{DR}}^2\right)(s + \lambda_R)(s + \lambda_S) \tag{1}$$

where subscripts DR, R, and S refer to the Dutch-roll, roll, and spiral modes, respectively. Letting

$$P_{\text{lat}}(s) = s^4 + Bs^3 + Cs^2 + Ds + E \tag{2}$$

one has

$$B = \lambda_R + \lambda_S + 2\zeta_{\text{DR}}\omega_{\text{DR}} \tag{3}$$

$$C = \omega_{\text{DR}}^2 + 2\zeta_{\text{DR}}\omega_{\text{DR}}(\lambda_R + \lambda_S) + \lambda_R\lambda_S \tag{4}$$

$$D = (\lambda_R + \lambda_S)\omega_{\text{DR}}^2 + 2\zeta_{\text{DR}}\omega_{\text{DR}}\lambda_R\lambda_S \tag{5}$$

$$E = \lambda_R\lambda_S\omega_{\text{DR}}^2 \tag{6}$$

Clearly, the polynomial coefficients B, C, D, and E depend on the aerodynamic derivatives. Assume a straight, level flight condition, and a stability axis reference frame. The lateral–directional system matrix is given by (Ref. 3, p. 354)

$$A_{\text{lat}} = \begin{bmatrix} Y_\beta/V_0 & 0 & -1 & g/V_0 \\ L'_\beta & L'_p & L'_r & 0 \\ N'_\beta & N'_p & N'_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{7}$$

and the corresponding coefficients of the fourth-order lateral characteristic polynomial are

$$B = -L'_p - N'_r - Y_\beta/V_0 \tag{8}$$

$$C = N'_\beta + L'_p(N'_r + Y_\beta/V_0) - N'_pL'_r + Y_\beta N'_r/V_0 \tag{9}$$

$$D = -L'_p(N'_\beta + Y_\beta N'_r/V_0) + N'_p(L'_\beta + Y_\beta L'_r/V_0) - L'_\beta g/V_0 \tag{10}$$

$$E = (L'_\beta N'_r - N'_\beta L'_r)g/V_0 \tag{11}$$

where L'_i and N'_i are the standard primed derivatives, g is the gravitational acceleration, and V_0 is the trim aircraft velocity. First consider the spiral mode approximation. Because the spiral is a first-order mode with a long time constant, it is customary to assume $|\lambda_S| \ll \min(|\lambda_R|, \omega_{\text{DR}})$, from which $D \cong \lambda_R\omega_{\text{DR}}^2$ and

$$\lambda_S \cong E/D \tag{12}$$

This is a classic result (Ref. 3, p. 378), which is known to give a suitable approximation to the spiral mode. Consider now the third-order polynomial,

$$\Delta(s) \triangleq s^3 + Bs^2 + Cs + D \tag{13}$$

Received 20 November 2003; revision received 15 March 2004; accepted for publication 15 March 2004.

*Associate Professor, Department of Aerospace Engineering; g.mengali@ing.unipi.it.

†Postdoctoral Fellow, Department of Electronics, Computer Science and Systems; fabrizio.giulietti@mail.ingfo.unibo.it. Member AIAA.

As long as the condition $|\lambda_S| \ll \min(|\lambda_R|, \omega_{\mathrm{DR}})$ is satisfied, one has

$$\Delta(s)(s + E/D) \cong P_{\mathrm{lat}}(s) \tag{14}$$

from which

$$s^3 + Bs^2 + Cs + D = \left(s^2 + 2\zeta_{\mathrm{DR}}\omega_{\mathrm{DR}}s + \omega_{\mathrm{DR}}^2\right)(s + \lambda_R) \tag{15}$$

This amounts to saying that the roll and Dutch-roll modes may be computed from Eq. (1) by simply setting λ_S to zero. To obtain a set of approximate analytical solutions to the roots of $\Delta(s)$, we use of the following result.

Proposition. Suppose that equation $\Delta(s) = 0$ has one real and two complex roots, that is, $s^3 + Bs^2 + Cs + D = (s^2 + 2\zeta_{\mathrm{DR}}\omega_{\mathrm{DR}}s + \omega_{\mathrm{DR}}^2)(s + \lambda_R)$, and assume $D \neq 0$. Then

$$B^2 - (C^2/D)\lambda_R = \left(1 - 4\zeta_{\mathrm{DR}}^2\right)(B\lambda_R - C) \tag{16}$$

Proof. Consider Eq. (15) and assume $D \neq 0$. Then

$$B = \lambda_R + 2\zeta_{\mathrm{DR}}\omega_{\mathrm{DR}} \tag{17}$$

$$C = \omega_{\mathrm{DR}}^2 + 2\zeta_{\mathrm{DR}}\omega_{\mathrm{DR}}\lambda_R \tag{18}$$

$$D = \lambda_R\omega_{\mathrm{DR}}^2 \tag{19}$$

Multiplying Eq. (17) by λ_R and Eq. (18) by -1 and summing yields

$$(\lambda_R - \omega_{\mathrm{DR}})(\lambda_R + \omega_{\mathrm{DR}}) = B\lambda_R - C \tag{20}$$

In passing observe that for $\lambda_R = \omega_{\mathrm{DR}}$ one has $\lambda_R = C/B$. Now divide Eq. (18) by ω_{DR} and combine the result (summing and subtracting) with Eq. (17) to produce

$$(\lambda_R - \omega_{\mathrm{DR}})(1 - 2\zeta_{\mathrm{DR}}) = B - C/\omega_{\mathrm{DR}} \tag{21}$$

$$(\lambda_R + \omega_{\mathrm{DR}})(1 + 2\zeta_{\mathrm{DR}}) = B + C/\omega_{\mathrm{DR}} \tag{22}$$

Multiplying Eqs. (21) and (22) term-by-term and making use of Eq. (20) gives

$$B^2 - C^2\big/\omega_{\mathrm{DR}}^2 = (B\lambda_R - C)\left(1 - 4\zeta_{\mathrm{DR}}^2\right) \tag{23}$$

from which the proof of Eq. (16) follows, recalling that $1/\omega_{\mathrm{DR}}^2 = \lambda_R/D$ [see Eq. (19)]. □

Equation (16) expresses a direct link between the real root, the damping ratio of the complex roots, and the coefficients of the polynomial. This result has interesting consequences when applied to the aircraft lateral–directional modes. In fact, because the Dutch roll is typically a lightly damped oscillatory mode, a reasonable approximation is to consider $4\zeta_{\mathrm{DR}}^2 \ll 1$. Then, from Eq. (16), the roll mode is approximated as

$$\lambda_R \cong (B^2 + C)/(B + C^2/D) \tag{24}$$

Also, the Dutch-roll mode is given by

$$2\zeta_{\mathrm{DR}}\omega_{\mathrm{DR}} = B - \lambda_R \cong C(BC - D)/(C^2 + BD) \tag{25}$$

$$\omega_{\mathrm{DR}}^2 = D/\lambda_R \cong (C^2 + BD)/(B^2 + C) \tag{26}$$

The accuracy of such approximations with respect to the exact solutions [the roots of $\Delta(s)$] is shown in Fig. 1 for different values of ζ_{DR}. Note that as long as $0.5 \leq \omega_{\mathrm{DR}}/\lambda_R \leq 2$, all of the three approximations give good estimates of the lateral–directional modes. Also note that for $\lambda_R = \omega_{\mathrm{DR}}$ the errors between the exact and the approximate results vanish, independently of the value of ζ_{DR}. In fact, for $\lambda_R = \omega_{\mathrm{DR}}$, one has $\lambda_R = C/B$ [Eq. (20)] and Eq. (16) becomes an identity.

Literal Approximations

Equations (24–26) may be easily translated into literal expressions involving aerodynamic derivatives through Eqs. (8–10). However, the corresponding approximations for the roll and Dutch-roll poles are quite involved. To obtain simpler expressions suppose that 1) $|Y_\beta N_r'/V_0| \ll |N_\beta'|$, 2) $|N_p' L_r'| \ll |N_\beta'|$, and 3) $|Y_\beta L_r'/V_0| \ll |L_\beta'|$. These are reasonable assumptions for a wide range of aircraft configurations and flight conditions (Ref. 3, p. 402). Accordingly, Eq. (24) yields

$$\lambda_R = \frac{(L_p' + N_r' + Y_v)^2 + N_\beta' + L_p'(N_r' + Y_v)}{-L_p' - N_r' - Y_v + [N_\beta' + L_p'(N_r' + Y_v)]^2\big/\left(N_\beta'\lambda_R^{(C)}\right)} \tag{27}$$

where $Y_v = Y_\beta/V_0$ and

$$\lambda_R^{(C)} = -L_p' + (L_\beta'/N_\beta')(N_p' - g/V_0) \tag{28}$$

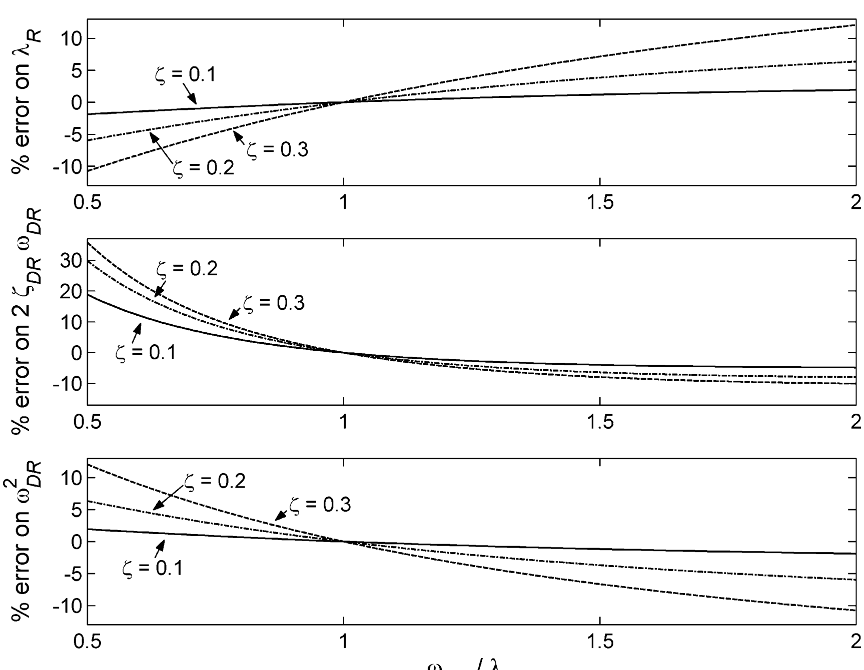

Fig. 1 Accuracy analysis of the roll mode, Dutch-roll damping, and Dutch-roll frequency approximations.

is the classic roll approximation, as indicated by the superscript (C), that is found by considering a spiral-roll subsidence model (McRuer et al.,[3] pp. 374–376). Also

$$2\zeta_{\text{DR}}\omega_{\text{DR}} = -(N_r' + Y_v) - (L_p' + \lambda_R) \tag{29}$$

$$\omega_{\text{DR}}^2 = N_\beta'\left(\lambda_R^{(C)}/\lambda_R\right) \tag{30}$$

where λ_R is given by Eq. (27). The expression for the Dutch-roll damping, given by Eq. (29), is the sum of two contributions: The first one, $-(N_r' + Y_v)$, coincides with the damping approximation of a two-degree-of-freedom Dutch-roll model; the second, $-(L_p' + \lambda_R)$, is the correction due to the roll mode. Note that L_p' is the roll subsidence pole obtained with a single-degree-of-freedom roll approximation. Because the Dutch roll is a lightly damped oscillatory mode, the precision of the numerical value obtained through Eq. (29) is strictly connected with the accuracy of the estimate of the roll mode, which is guaranteed by Eq. (24). Finally, note that the classic approximation for the Dutch-roll natural frequency $\omega_{\text{DR}}^2 = N_\beta'$ is only slightly modified by Eq. (30) because $\lambda_R^{(C)} \approx \lambda_R$.

To validate the proposed approximations, extensive numerical simulations have been performed using aircraft data taken from the literature.[4] In particular, it is interesting to quantify the order of magnitude of the errors due to the Dutch-roll damping approximation. Table 1 shows an example of the literal approximation error analysis carried out for a number of data and different aircraft configurations. The second column (from the left) lists the exact numerical value of $2\zeta_{\text{DR}}\omega_{\text{DR}}$. The third column provides the percent errors between the numerical values of the Dutch-roll damping obtained using a classic two-degree-of-freedom Dutch-roll model (McRuer et al.,[3] p. 367) and their exact values. The fourth and fifth columns provide the errors associated with the full and simplified approximations given by Eqs. (25) and (29). Note that for all of the flight conditions of Table 1 the errors obtained with both the full and the simplified approximation are less than 10%, with remarkable improvements with respect to the classic simplified formula. Because the traditional approximations are known to yield reasonable results for λ_R and ω_{DR}, a numerical comparison with the new literal expressions is not reproduced here. However, note that the percent errors for λ_R and ω_{DR} are less than 5 and 1.5%, respectively [using the simplified formula (29)] for all of the flight conditions reported in Table 1. The same order of magnitude for the errors is obtained when power approach conditions are considered. This latter result contrasts with other expressions found in the literature,[5] where not as good approximations are obtained with low-speed flight conditions.

Table 1 Comparison between different approximations of Dutch-roll damping

Flight condition[a]	$2\zeta_{\text{DR}}\omega_{\text{DR}}$[b]	Percent error e_1[c]	Percent error e_2[d]	Percent error e_3[e]
		F-104A		
3	1.1056	78.4	0.7	−9.8
4	1.2200	−25.4	−4.6	−2.1
5	0.0781	395.1	0.0	−2.7
6	0.0346	951.0	7.9	−8.1
7	0.3901	59.8	−2.1	−3.2
		Boeing 747		
3	0.2673	52.7	6.8	4.6
4	0.4314	25.4	4.6	4.2
5	0.1200	121.1	1.3	−2.6
6	0.1769	74.9	0.4	−2.5
7	0.2572	36.8	−1.2	−2.1
		Convair 880M		
3	0.3173	9.4	−2.2	−0.1
4	0.4978	4.1	−4.6	−4.0
5	0.2791	−1.4	−4.2	−1.7
6	0.2585	10.9	−3.4	−1.9
7	0.2866	7.3	−3.8	−2.6

[a]Data and flight conditions from Heffley and Jewell.[4]
[b]Exact value.
[c]Percent error with a classic two-degree-of-freedom Dutch-roll model.
[d]Percent error with full approximation given by Eq. (25).
[e]Percent error with simplified approximation given by Eq. (29).

Departure Criteria

We now turn our attention to literal approximations of lateral–directional departure criteria. For the fourth-order characteristic equation $P_{\text{lat}}(s) = 0$, the necessary and sufficient conditions for stability are $B > 0$, $BC - D > 0$, $R > 0$, and $E > 0$, where

$$R = BCD - EB^2 - D^2 \tag{31}$$

is the Routh's discriminant. The inequality $E > 0$ is commonly referred to as the generalized static stability criterion for the lateral–directional motion.[2] A violation of this stability condition is caused by the presence of a real pole in the right half-plane and reveals a spiral mode instability. However, another form of instability is possible. In fact, Duncan[6] has shown that the real part of a conjugate complex pair of roots vanishes if (and only if) $R = 0$. This is the mathematical condition for the occurrence of a Dutch-roll instability.

Substituting the aerodynamic derivatives into Eq. (31) gives an expression of the literal departure criteria for the Dutch-roll mode that is too involved to be of practical use. Alternatives to this approach are theoretically offered by setting to zero the literal expression of the Dutch-roll damping. However, it is well known that existing analytical approximations are unable to predict the occurrence of a Dutch-roll instability, due to their scanty or insufficient accuracy. To solve this problem, we distinguish between two cases: 1) There exist distinct timescales between a slow spiral mode and fast roll and Dutch-roll modes, and 2) there is no clear distinction between fast and slow modes. These cases generally correspond to conventional aircraft at low and high angle of attack (AOA) flight, respectively.

Departure Criteria for Low AOA Flight

In this case, the condition $|\lambda_S| \ll \min(|\lambda_R|, \omega_{\text{DR}})$ is satisfied. Accordingly, Routh's discriminant may be simplified as

$$R = B^2D(C/B - E/D - D/B^2) \cong B^2D(C/B - D/B^2) \tag{32}$$

where Eq. (12) has been employed. In other words, the condition for Dutch-roll stability reduces to $BC - D > 0$. Note that this is exactly the stability condition implied by Eq. (25). As a result, the approximate literal expression given by Eq. (29) effectively predicts the Dutch-roll instability. This may also be verified by simulation. Consider, for instance, the aircraft data representative of the F-104A in a power approach configuration.[4] The Dutch-roll mode is unstable, with $\zeta_{\text{DR}} = -0.034$ and $2\zeta_{\text{DR}}\omega_{\text{DR}} = -0.144$ rad/s. Both the full and simplified approximations (25) and (29) yield excellent estimates of the Dutch-roll damping coefficient $2\zeta_{\text{DR}}\omega_{\text{DR}}$, with errors equal to 1.0 and 11.8%, respectively. Note that the classic two-degree-of-freedom Dutch-roll model is unable to predict this instability.

Departure Criteria for High AOA Flight

The demand for an air superiority role requires modern aircraft to fly at high AOA. Therefore, analytical criteria to prevent aircraft departure from controlled flight have been studied over the past few years.[2,7] In particular, in a recent paper Ananthkrishnan et al.[8] have proposed a new method to predict the occurrence of a Dutch-roll instability. With use of the fact that at high AOA the four lateral–directional modes usually occur as two couples of complex roots with small damping ratio, in Ref. 8 the lightly damped lateral dynamics are replaced by a Hamiltonian system, and an analytical condition for the occurrence of a Hamiltonian Hopf bifurcation is employed to establish the departure criteria.

We now reconsider the same problem with a sharply different methodology, using an algebraic approach. To proceed, first observe that the assumption of four complex poles implies that

$$P_{\text{lat}}(s) = \left(s^2 + 2\zeta_{\text{DR}}\omega_{\text{DR}}s + \omega_{\text{DR}}^2\right)\left(s^2 + 2\zeta_{\text{SR}}\omega_{\text{SR}}s + \omega_{\text{SR}}^2\right) \tag{33}$$

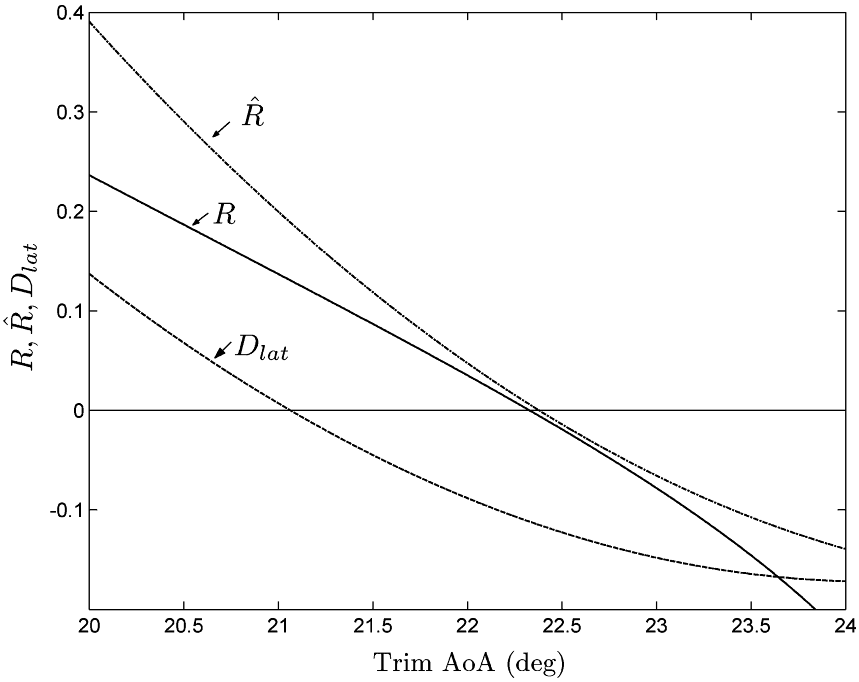

Fig. 2 Comparison between different predictions of Dutch-roll instability through Routh's discriminant R, the approximate Routh's discriminant $\hat{R}$, and the D_{lat} criterion.[8]

where subscript SR denotes the spiral-roll complex poles. Assume that both the Dutch-roll and the spiral-roll modes are very lightly damped, that is, $\zeta_{\mathrm{DR}} \cong \zeta_{\mathrm{SR}} \cong 0$. Then

$$P_{\mathrm{lat}}(s) \cong s^4 + Cs^2 + E \tag{34}$$

The necessary and sufficient conditions for dynamic stability are found applying Routh's criterion to Eq. (34). The Routh array for this polynomial is

$$\begin{array}{lccc} s^4: & 1 & C & E \\ s^3: & 4 & 2C & 0 \\ s^2: & C/2 & E & 0 \\ s^1: & (C^2-4E)/C & 0 & 0 \\ s^0: & E & 0 & 0 \end{array}$$

Note that a special case of Routh's test occurs because the second row, being entirely zero, has been replaced by the coefficients obtained from the derivative of the auxiliary polynomial.[9] This latter is formed from the coefficient of the first row and, therefore, coincides with $P_{\mathrm{lat}}(s)$. Accordingly, the second row of the Routh array contains the coefficients of $\mathrm{d}P_{\mathrm{lat}}(s)/\mathrm{d}s = 4s^3 + 2Cs$.

Recall that necessary and sufficient conditions for stability are that all of the elements in the first column of the Routh array are positive, then the stability condition is simply

$$\hat{R} \triangleq C^2 - 4E > 0 \tag{35}$$

because $C \cong \omega_{\mathrm{DR}}^2 + \omega_{\mathrm{SR}}^2 > 0$ and $E = \omega_{\mathrm{DR}}^2 \omega_{\mathrm{SR}}^2 > 0$. In other words, $\hat{R} > 0$ replaces the exact condition $R > 0$, where R is given by Eq. (31). Note that the inequality (35) looks similar but is substantially different from the D_{lat} criterion proposed by Ananthkrishnan et al.[8] In fact, this latter is in the form $D_{\mathrm{lat}} = Q^2 - 4S_{\mathrm{lat}} > 0$, which has the same structure as $\hat{R}$, and $S_{\mathrm{lat}} = E$, but $Q \neq C$. The reason for this difference may be briefly explained as follows. In their paper, Ananthkrishnan et al.[8] describe the linearized lateral dynamics in second-order form as $\ddot{\boldsymbol{y}}_{\mathrm{lat}} + C_{\mathrm{lat}}\dot{\boldsymbol{y}}_{\mathrm{lat}} + K_{\mathrm{lat}}\boldsymbol{y}_{\mathrm{lat}} = 0$ where C_{lat} and K_{lat} are suitable 2×2 damping and stiffness matrices. Then, the undamped approximation is simply obtained by ignoring the damping matrix C_{lat}. However, it may be shown that this is a sufficient, but not necessary condition to guarantee the eigenvalues of the linearized dynamics to be imaginary. This contrasts with our methodology, which provides necessary and sufficient conditions to the problem solution. Actually, the differences between the two approaches are remarkable both from a theoretical and a practical viewpoint. This conclusion may be verified by a numerical example. For illustrative purposes we use the model aircraft data taken from Fan et al.[10] The eigenvalues of the linearized dynamics around a straight, level flight condition for different trim AOA show that a Dutch-roll instability occurs at a trim AOA around 22.5 deg. This is confirmed by the simulation shown in Fig. 2. Note that the approximate result predicted by $\hat{R}$ is almost equal to the exact value given by Routh's discriminant and that $\hat{R}$ gives much more accurate information than the value predicted by the D_{lat} criterion.

Conclusions

A unified algebraic approach has been developed to approximate aircraft lateral–directional modes and predict departure susceptibility. Under the assumption of a long time constant spiral mode, the complete fourth-order characteristic polynomial has been first factorized as the product of a first- and a third-order polynomials. The root of the first one is shown to coincide with the classic spiral mode approximation ($\lambda_S \cong E/D$). The approximations for the roots of the third-order polynomial come from an interesting factorization that allows one to relate the real root, the coefficients of the polynomial, and the damping ratio of the complex roots, but does not contain their natural frequency. The polynomial roots coincide with the roll and Dutch-roll eigenvalues. As a consequence of the particular factorization employed, an approximate literal solution is found by simply assuming a lightly damped Dutch-roll pole. The corresponding roll mode approximation is then used to establish compact expression for the Dutch-roll natural frequency and damping. These approximations may be further simplified by neglecting some aerodynamic derivatives that common practice has shown to be relatively unimportant. The Dutch-roll damping approximation accurately predicts the occurrence of an aircraft departure because the stability condition of the approximate expression is equivalent to that implied by Routh's criterion. Finally, an analytical criterion for aircraft departure in high AOA conditions has been developed. The proposed expression looks similar, but is significantly different and more accurate than the D_{lat} criterion that has been recently suggested[8] to address the same problem. Extensive simulations comprising both the approximations for the lateral–directional modes and the expressions for the departure criteria have been included in the paper. They confirm the effectiveness of the proposed approach, which may be successfully employed early in the design process.

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Limit Cycles and Domain of Stability in Unsteady Aeroelastic System

Sushma Gujjula,* Sahjendra N. Singh,† and Woosoon Yim‡

University of Nevada, Las Vegas, Nevada 89154-4026

Introduction

AEROELASTIC systems exhibit a variety of phenomena including instability, limit-cycle oscillation (LCO), and even chaotic vibration.[1] An excellent survey paper by Mukhopadhyay[2] provides a historical perspective on analysis and control of aeroelastic systems. Studies related to prediction of flutter instability theory have been done.[1,3–8] In this Note, the existence of LCO and the domain of stability (attraction) in prototypical aeroelastic wing sections are examined. Unlike Ref. 8, the aeroelastic model considered here includes the unsteady aerodynamics based on Theodorsen's theory.[9] The model includes a structural nonlinearity of fifth degree in the pitch degree of freedom. The chosen dynamic model describes the nonlinear plunge and pitch motion of a wing.[4,5] The dual-input describing functions (DIDFs)[10] of the nonlinearity for asymmetric oscillations are used for the prediction of unstable and stable limit cycles. For the case when the origin is stable, the quadratic Lyapunov method (see Ref. 11) is used to compute the domain of attraction. Analytical expressions for the computation of biases, amplitudes, and frequencies of oscillation of the pitch and plunge responses are obtained. The orbital stability of the LCO using the Nyquist criterion is established, and it is shown that both unstable as well as stable limit cycles exist when the origin is exponentially stable.

Received 18 December 2003; revision received 19 February 2004; accepted for publication 20 February 2004.

*Graduate Student, Electrical and Computer Engineering.

†Professor, Electrical and Computer Engineering.

‡Professor, Mechanical Engineering.

Aeroelastic Model

The prototypical aeroelastic wing section is shown in Fig. 1. The governing equations of motion of the aeroelastic system are given by[4,5]

$$\begin{bmatrix} m_t & m_w x_\alpha b \\ m_w x_\alpha b & I_\alpha \end{bmatrix}\begin{bmatrix} \ddot{h} \\ \ddot{\alpha} \end{bmatrix} + \begin{bmatrix} c_h & 0 \\ 0 & c_\alpha \end{bmatrix}\begin{bmatrix} \dot{h} \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} k_h & 0 \\ 0 & k_\alpha(\alpha) \end{bmatrix}\begin{bmatrix} h \\ \alpha \end{bmatrix} = \begin{bmatrix} -L(t) \\ M(t) \end{bmatrix} \tag{1}$$

where α is the pitch angle, b is the semichord of the wing, h is the plunge displacement, m_w and m_t are the mass of the wing and the total mass, I_α is the moment of inertia, and x_α is the nondimensionalized distance of the center of mass from the elastic axis. The lift $L(t)$ and moment $M(t)$ represent the unsteady aerodynamics, which are functions of position, velocity, acceleration, and time prehistory of the vortex wave. The lift and moment are acting at the elastic axis of the wing. Here c_α and c_h are the pitch and plunge damping coefficients, k_{h0} is the plunge spring constant, and $k_\alpha(\alpha)$ is the nonlinear function associated with the pitch spring. For purposes of illustration, the function $\alpha k_\alpha(\alpha)$ is considered as a polynomial nonlinearity of fifth degree. This is given by $\alpha k_\alpha(\alpha) = \alpha(k_{\alpha_0} + k_{\alpha_1}\alpha + k_{\alpha_2}\alpha^2 + k_{\alpha_3}\alpha^3 + k_{\alpha_4}\alpha^4)$.

Theodorsen[9] derived the expressions for lift and moment, assuming harmonic motion of the airfoil, of the form[5]

$$\begin{aligned} -L(t) &= -\rho b^2 s_p(U\pi\dot{\alpha} + \pi\ddot{h} - \pi b a\ddot{\alpha}) \\ &\quad - 2\pi\rho U b s_p C(k)\big[U\alpha + \dot{h} + b\big(\tfrac{1}{2} - a\big)\dot{\alpha}\big] \\ M(t) &= -\rho b^2 s_p\Big\{\pi\big(\tfrac{1}{2} - a\big)U b\dot{\alpha} + \pi b^2\big(\tfrac{1}{8} + a^2\big)\ddot{\alpha} - a\pi b\ddot{h}\Big\} \\ &\quad + 2\rho U b^2\pi s_p\big(\tfrac{1}{2} + a\big)C(k)\big[U\alpha + \dot{h} + b\big(\tfrac{1}{2} - a\big)\dot{\alpha}\big] \end{aligned} \tag{2}$$

where s_p is the span and a is nondimensionalized distance from the midchord to the elastic axis and U is freestream velocity. Jones

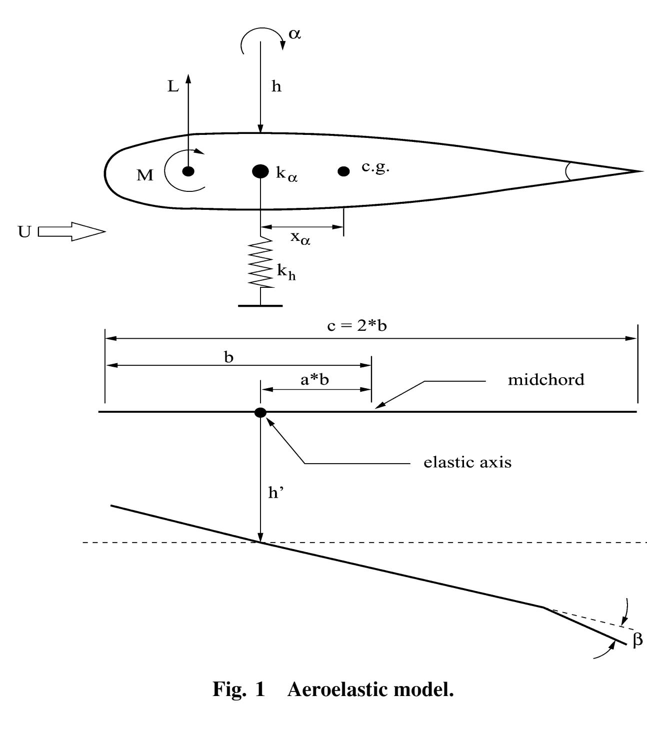

Fig. 1 Aeroelastic model.